International Journal of Mathematics, Statistics and Operations Research
Vol. 5; Number 1; 2025, pp. 105-136
https://doi.org/10.47509/IJMSOR.2025.v05i01.07

ARF INDIA
Academic Open Access Publishing
www.arfjournals.com

On Generalized co-Padovan Numbers

Yüksel Soykan[a]

[a] *Department of Mathematics, Faculty of Science, Zonguldak Bülent Ecevit University, Zonguldak, Türkiye*

ABSTRACT
In this paper, we introduce and investgate a new third order recurrence sequence so called generalized co-Padovan sequence and its two special subsequences which are related to generalized padovan numbers and its two subsequences. There are close interrelations between recurrence equations of and roots of characteristic equations of generalized Padovan and generalized co-Padovan numbers. We present Binet's formulas, generating functions, some identities, Simson's formulas, recurrence properties, sum formulas and matrices related with these sequences.

2010 Mathematics Subject Classification. 11B37, 11B39, 11B83.

KEYWORDS
Padavon numbers, Perrin numbers, co-Padovan numbers, co-Perrin numbers, third order recurrence relations, Binet's formula, generating functions.

1. Introduction: Generalized Padovan Numbers

The generalized Tribonacci numbers

$$\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$$

(or $\{W_n\}_{n\geq 0}$ or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \qquad W_0 = a, W_1 = b, W_2 = c, \quad n \geq 3 \tag{1.1}$$

where $W_0, W_1, W_2$are arbitrary complex (or real) numbers and r, s and t are real numbers with $t \neq 0$.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.1) holds for all integers n.

CONTACT Author[a]. Email:yukesel$_{s}oykan@hotmail.com, yuksel.soykan@gmail.com$

Article History
Received : 22 March 2025; Revised : 20 April 2025; Accepted : 11 May 2025; Published : 26 June 2025

To cite this paper
Yüksel Soykan (2025). On Generalized co-Padovan Numbers. *International Journal of Mathematics, Statistics and Operations Research.* 5(1), 105-136.

For r, s, t satisfying Eq. (1.1), the generalized co-Tribonacci numbers

$$\{Y_n(Y_0, Y_1, Y_2; -s, -rt, t^2)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -sY_{n-1} - rtY_{n-2} + t^2Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{1.2}$$

i.e.,

$$Y_n = r_1Y_{n-1} + s_1Y_{n-2} + t_1Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3$$

where $Y_0, Y_1, Y_2$are arbitrary complex (or real) numbers and $r_1 = -s$, $s_1 = -rt$, $t_1 = t^2$.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} Y_{-n} &= -\frac{-rt}{t^2}Y_{-(n-1)} - \frac{-s}{t^2}Y_{-(n-2)} + \frac{1}{t^2}Y_{-(n-3)} \\ &= -\frac{s_1}{t_1}Y_{-(n-1)} - \frac{r_1}{t_1}Y_{-(n-2)} + \frac{1}{t_1}Y_{-(n-3)} \end{aligned}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1.2) holds for all integer n. For more information on generalized Tribonacci and co-Tribonacci numbers, see [4].

Note that we can easily use and modify the results given for r, s, t in [4] by substituting r_1, s_1, t_1 for r, s, t and we will do this in this paper.

There are close interrelations between roots of characteristic equations of generalized Tribonacci and generalized co-Tribonacci numbers, see [4, Lemma 17.]: If α, β, γ are the roots of characteristic equation of $\{W_n\}$ which is given as

$$z^3 - rz^2 - sz - t = 0,$$

and if $\theta_1, \theta_2, \theta_3$ are the roots of characteristic equation of $\{Y_n\}$ which is given as

$$y^3 - r_1y^2 - s_1y - t_1 = y^3 + sy^2 + rty - t^2 = 0,$$

then we get

$$\begin{aligned} \theta_1 &= \beta\gamma, \\ \theta_2 &= \alpha\beta, \\ \theta_3 &= \alpha\gamma. \end{aligned}$$

There are also close connections and relations between recurrence equations of generalized Tribonacci and generalized co-Tribonacci numbers, see, for example, Lemma 32 in [4].

In this paper, we consider the case $r = 0$, $s = 1$, $t = 1$ so that $r_1 = -s = -1$, $s_1 = -rt = 0$, $t_1 = t^2 = 1$.

In the next section, we also use the notation $r = -1$, $s = 0$, $t = 1$ for $r_1 = -1$, $s_1 = 0$, $t_1 = 1$ to use results in the paper [4]. Now, in this section, for the case $r = 0$, $s = 1$, $t = 1$ we present some well known results.

A generalized Padovan sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = W_{n-2} + W_{n-3} \tag{1.3}$$

with the initial values $W_0 = c_0, W_1 = c_1, W_2 = c_2$ not all being zero.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -W_{-(n-1)} + W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.3) holds for all integer n.

As $\{W_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$z^3 - z - 1 = 0 = (z - \alpha)(z - \beta)(z - \gamma) = 0.$$

The roots α, β, γ of characteristic equation of $\{W_n\}$ are given as

$$\begin{aligned}
\alpha &= \left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} = 1.324\,717\,957\,24 \\
\beta &= \omega\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3} \\
\gamma &= \omega^2\left(\frac{1}{2} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{1}{2} - \sqrt{\frac{23}{108}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that there are the following relations between the roots of characteristic equation:

$$\begin{aligned}
\alpha + \beta + \gamma &= 0, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= -1, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The sequence $\{W_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of W_n, Binet's formula of W_n can be given as follows:

Theorem 1.1. *For all integers n, Binet's formula of generalized Padovan numbers is given as follows.*

$$\begin{aligned}
W_n &= \frac{p_1\alpha^n}{(\alpha - \beta)(\alpha - \gamma)} + \frac{p_2\beta^n}{(\beta - \alpha)(\beta - \gamma)} + \frac{p_3\gamma^n}{(\gamma - \alpha)(\gamma - \beta)} \\
&= A_1\alpha^n + A_2\beta^n + A_3\gamma^n,
\end{aligned}$$

where

$$\begin{aligned} p_1 &= W_2-(\beta+\gamma)W_1+\beta\gamma W_0, && (1.4)\\ p_2 &= W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0, && (1.5)\\ p_3 &= W_2-(\alpha+\beta)W_1+\alpha\beta W_0 && (1.6) \end{aligned}$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\alpha-\beta)(\alpha-\gamma)}=\frac{W_2-(\beta+\gamma)W_1+\beta\gamma W_0}{(\alpha-\beta)(\alpha-\gamma)}\\ &= \frac{(\alpha W_2+\alpha^2W_1+W_0)}{2\alpha+3},\\ A_2 &= \frac{p_2}{(\beta-\alpha)(\beta-\gamma)}=\frac{W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0}{(\beta-\alpha)(\beta-\gamma)}\\ &= \frac{(\beta W_2+\beta^2W_1+W_0)}{2\beta+3},\\ A_3 &= \frac{p_3}{(\gamma-\alpha)(\gamma-\beta)}=\frac{W_2-(\alpha+\beta)W_1+\alpha\beta W_0}{(\gamma-\alpha)(\gamma-\beta)}\\ &= \frac{(\gamma W_2+\gamma^2W_1+W_0)}{2\gamma+3} \end{aligned}$$

Proof. For the proof, take $r=0$, $s=1$, $t=1$ in [4, Theorem 3 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} W_nz^n$ of the sequence W_n.

Lemma 1.1.1. *Suppose that $f_{W_n}(z)=\sum_{n=0}^{\infty} W_nz^n$ is the ordinary generating function of the generalized Padovan numbers $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_nz^n$ is given by*

$$\sum_{n=0}^{\infty} W_nz^n=\frac{W_0+W_1z+(W_2-W_0)z^2}{1-z^2-z^3}. \qquad (1.7)$$

Proof. Set $r=0$, $s=1$, $t=1$ in [4, Lemma 9.]. □

Now we define three special cases of the sequence $\{W_n\}$. adjusted Padovan sequence $\{U_n\}_{n\geq 0}$ (a variant of the sequence $\{P_n\}$), Perrin (Padovan-Lucas) sequence $\{E_n\}_{n\geq 0}$ (OEIS: A001608, [2]) and Padovan sequence $\{P_n\}_{n\geq 0}$ (OEIS: A000931, [2]) are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} U_{n+3} &= U_{n+1}+U_n, \qquad U_0=0,U_1=1,U_2=0, && (1.8)\\ E_{n+3} &= E_{n+1}+E_n, \qquad E_0=3,E_1=0,E_2=2, && (1.9)\\ P_{n+3} &= P_{n+1}+P_n, \qquad P_0=1,P_1=1,P_2=1. && (1.10) \end{aligned}$$

The sequences $\{U_n\}_{n\geq 0}$, $\{E_n\}_{n\geq 0}$ and $\{P_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} U_{-n} &= -U_{-(n-1)} + U_{-(n-3)}, \\ E_{-n} &= -E_{-(n-1)} + E_{-(n-3)}, \\ P_{-n} &= -P_{-(n-1)} + P_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.8)-(1.10) hold for all integer n. There are several studying on Padovan numbers, see for example [1, 3] and references thereafter.

For all integers n, Binet's formula of adjusted Padovan, Perrin (Padovan-Lucas) and Padovan numbers (using initial conditions (1.4)-(1.6) in Theorem 1.1) can be expressed as follows:

Theorem 1.2. *For all integers n, Binet's formulas of adjusted Padovan, Perrin (Padovan-Lucas) and Padovan numbers are*

$$\begin{aligned} U_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)} \\ &= \frac{\alpha^{n+2}}{2\alpha+3} + \frac{\beta^{n+2}}{2\beta+3} + \frac{\gamma^{n+2}}{2\gamma+3}, \\ E_n &= \alpha^n + \beta^n + \gamma^n, \\ P_n &= \frac{\alpha^{n+4}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+4}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+4}}{(\gamma-\alpha)(\gamma-\beta)}, \end{aligned}$$

respectively.

Lemma 1.1.1 gives the following results as particular examples (generating functions of adjusted Padovan, Perrin (Padovan-Lucas) and Padovan numbers).

Corollary 1.3. *Generating functions of adjusted Padovan, Perrin (Padovan-Lucas) and Padovan numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} U_n z^n &= \frac{z}{1-z^2-z^3}, \\ \sum_{n=0}^{\infty} E_n z^n &= \frac{3-z^2}{1-z^2-z^3}, \\ \sum_{n=0}^{\infty} P_n z^n &= \frac{1+z}{1-z^2-z^3}, \end{aligned}$$

respectively.

Now, we present some identities of adjusted Padovan, Perrin (Padovan-Lucas) and Padovan numbers. First, we can give a few basic relations between $\{U_n\}$ and $\{E_n\}$.

Lemma 1.3.1. *The following equalities are true:*

(a) $E_n = 2U_{n+4} + U_{n+3} - 3U_{n+2}$.
(b) $E_n = U_{n+3} - U_{n+2} + 2U_{n+1}$.
(c) $E_n = -U_{n+2} + 3U_{n+1} + U_n$.

(d) $E_n = 3U_{n+1} - U_{n-1}$.
(e) $E_n = 2U_{n-1} + 3U_{n-2}$.
(f) $23U_n = -6E_{n+3} + 4E_{n+4} + 5E_{n+2}$.
(g) $23U_n = -6E_{n+3} + 9E_{n+2} + 4E_{n+1}$.
(h) $23U_n = 9E_{n+2} - 2E_{n+1} - 6E_n$.
(i) $23U_n = -2E_{n+1} + 3E_n + 9E_{n-1}$.
(j) $23U_n = 3E_n + 7E_{n-1} - 2E_{n-2}$.

Next, we give a few basic relations between $\{U_n\}$ and $\{P_n\}$.

Lemma 1.3.2. *The following equalities are true:*

(a) $U_n = P_{n+2} - P_{n+1}$.
(b) $U_n = -P_{n+1} + P_n + P_{n-1}$.
(c) $U_n = P_n - P_{n-2}$.
(d) $P_n = U_{n+1} + U_n$.
(e) $P_n = U_{n+1} + U_n$.
(f) $P_n = U_n + U_{n-1} + U_{n-2}$.

Now, we present a few basic relations between $\{E_n\}$ and $\{P_n\}$.

Lemma 1.3.3. *The following equalities are true:*

(a) $E_n = -3P_{n+2} + 2P_{n+1} + 4P_n$.
(b) $E_n = 2P_{n+1} + P_n - 3P_{n-1}$.
(c) $E_n = P_n - P_{n-1} + 2P_{n-2}$.
(d) $23P_n = 7E_{n+2} + E_{n+1} + 3E_n$.
(e) $23P_n = E_{n+1} + 10E_n + 7E_{n-1}$.
(f) $23P_n = 10E_n + 8E_{n-1} + E_{n-2}$.

2. Generalized co-Padovan Numbers

If $r = 0$, $s = 1$, $t = 1$, then we get $r_1 = -1$, $s_1 = 0$, $t_1 = 1$. From now on, throughout the paper, we also use the notation $r = 0$, $s = 1$, $t = 1$ for $r_1 = -1$, $s_1 = 0$, $t_1 = 1$ and we consider the case $r = 0$, $s = 1$, $t = 1$ to use results in the paper [4].

In this section, we define and investigate a new sequence and its three special cases, namely the generalized co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers. The generalized co-Padovan numbers

$$\{Y_n(Y_0, Y_1, Y_2; -1, 0, 1)\}_{n\geq 0}$$

(or shortly $\{Y_n\}_{n\geq 0}$) is defined as follows:

$$Y_n = -Y_{n-1} + Y_{n-3}, \qquad Y_0 = d, Y_1 = e, Y_2 = f, \quad n \geq 3 \tag{2.1}$$

where Y_0, Y_1, Y_2 are arbitrary complex (or real) numbers with real coefficients.

The sequence $\{Y_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$Y_{-n} = Y_{-(n-2)} + Y_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (2.1) holds for all integer n.

The first few generalized co-Padovan numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized co-Padovan numbers

n	Y_n	Y_{-n}
0	Y_0	Y_0
1	Y_1	$Y_1 + Y_2$
2	Y_2	$Y_0 + Y_1$
3	$Y_0 - Y_2$	$Y_0 + Y_1 + Y_2$
4	$Y_1 - Y_0 + Y_2$	$Y_0 + 2Y_1 + Y_2$
5	$Y_0 - Y_1$	$2Y_0 + 2Y_1 + Y_2$
6	$Y_1 - Y_2$	$2Y_0 + 3Y_1 + 2Y_2$
7	$2Y_2 - Y_0$	$3Y_0 + 4Y_1 + 2Y_2$
8	$2Y_0 - Y_1 - 2Y_2$	$4Y_0 + 5Y_1 + 3Y_2$
9	$2Y_1 - 2Y_0 + Y_2$	$5Y_0 + 7Y_1 + 4Y_2$
10	$Y_0 - 2Y_1 + Y_2$	$7Y_0 + 9Y_1 + 5Y_2$
11	$Y_0 + Y_1 - 3Y_2$	$9Y_0 + 12Y_1 + 7Y_2$
12	$Y_1 - 3Y_0 + 4Y_2$	$12Y_0 + 16Y_1 + 9Y_2$
13	$4Y_0 - 3Y_1 - 3Y_2$	$16Y_0 + 21Y_1 + 12Y_2$

Remark 2.1. *In this paper we will extensively use the paper [4]. Note that in the notation of [4], here we have* $r = 0$, $s = 1$, $t = 1$ *and* $r_1 = -1$, $s_1 = 0$, $t_1 = 1$. *For simplicity, we can use the result of [4] by taking and replacing* $r = -1$, $s = 0$, $t = 1$.

As $\{Y_n\}$ is a third-order recurrence sequence (difference equation), it's characteristic equation (cubic equation) is

$$y^3 + y^2 - 1 = 0.$$

The roots of characteristic equation of $\{Y_n\}$ are

$$\begin{aligned}
\theta_1 &= \frac{-1}{3} + \left(\frac{25}{54} + \sqrt{\frac{23}{108}}\right)^{1/3} + \left(\frac{25}{54} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\
\theta_2 &= \frac{-1}{3} + \omega\left(\frac{25}{54} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega^2\left(\frac{25}{54} - \sqrt{\frac{23}{108}}\right)^{1/3}, \\
\theta_3 &= \frac{-1}{3} + \omega^2\left(\frac{25}{54} + \sqrt{\frac{23}{108}}\right)^{1/3} + \omega\left(\frac{25}{54} - \sqrt{\frac{23}{108}}\right)^{1/3},
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

There are the following relations between the roots of characteristic equation:

$$\begin{cases} \theta_1 + \theta_2 + \theta_3 = -1, \\ \theta_1\theta_2 + \theta_1\theta_3 + \theta_2\theta_3 = 0, \\ \theta_1\theta_2\theta_3 = 1. \end{cases}$$

Note that there are an important relation between $\theta_1, \theta_2, \theta_3$ and α, β, γ:

$$\theta_1 = \beta\gamma,$$
$$\theta_2 = \alpha\beta,$$
$$\theta_3 = \alpha\gamma.$$

The sequence $\{Y_n\}$ can be expressed with Binet's formula. Using the roots of characteristic equation and the recurrence relation of Y_n, Binet's formula of Y_n can be given as follows:

Theorem 2.2. *For all integers n, Binet's formula of generalized co-Padovan numbers is given as follows:*

$$\begin{aligned} Y_n &= \frac{p_1\theta_1^n}{(\theta_1-\theta_2)(\theta_1-\theta_3)} + \frac{p_2\theta_2^n}{(\theta_2-\theta_1)(\theta_2-\theta_3)} + \frac{p_3\theta_3^n}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= A_1\theta_1^n + A_2\theta_2^n + A_3\theta_3^n, \end{aligned}$$

where

$$p_1 = Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0,\ p_2 = Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0,\ p_3 = Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0,$$

and

$$\begin{aligned} A_1 &= \frac{p_1}{(\theta_1-\theta_2)(\theta_1-\theta_3)} = \frac{Y_2-(\theta_2+\theta_3)Y_1+\theta_2\theta_3Y_0}{(\theta_1-\theta_2)(\theta_1-\theta_3)} \\ &= \frac{(\theta_1Y_2+\theta_1(1+\theta_1)Y_1+Y_0)}{-\theta_1^2+3}, \\ A_2 &= \frac{p_2}{(\theta_2-\theta_1)(\theta_2-\theta_3)} = \frac{Y_2-(\theta_1+\theta_3)Y_1+\theta_1\theta_3Y_0}{(\theta_2-\theta_1)(\theta_2-\theta_3)} \\ &= \frac{(\theta_2Y_2+\theta_2(1+\theta_2)Y_1+Y_0)}{-\theta_2^2+3}, \\ A_3 &= \frac{p_3}{(\theta_3-\theta_1)(\theta_3-\theta_2)} = \frac{Y_2-(\theta_1+\theta_2)Y_1+\theta_1\theta_2Y_0}{(\theta_3-\theta_1)(\theta_3-\theta_2)} \\ &= \frac{(\theta_3Y_2+\theta_3(1+\theta_3)Y_1+Y_0)}{-\theta_3^2+3}. \end{aligned}$$

Proof. For the proof, take $r=-1$, $s=0$, $t=1$ in [4, Theorem 3 (a)] or $r=-1$, $s=0$, $t=1$ in [4, Theorem 19 (a)]. □

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} Y_n z^n$ of the sequence Y_n.

Lemma 2.2.1. *Suppose that $f_{Y_n}(z) = \sum_{n=0}^{\infty} Y_n z^n$ is the ordinary generating function of the generalized co-Padovan numbers $\{Y_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} Y_n z^n$ is given by*

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{Y_0+(Y_1+Y_0)z+(Y_2+Y_1)z^2}{1+z-z^3}.$$

Proof. Set $r=-1$, $s=0$, $t=1$ in [4, Lemma 9.] or $r_1=-1, s_1=0, t_1=1$ in [4, Lemma 24.]. □

In this paper, we define and investigate, in detail, three special cases of the generalized co-Padovan numbers $\{Y_n\}$ which we call them co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers. co-adjusted Padovan numbers $\{K_n\}_{n\geq 0}$, co-Perrin (co-Padovan-Lucas) numbers $\{S_n\}_{n\geq 0}$ and co-Padovan numbers numbers $\{R_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$\begin{aligned} K_{n+3} &= -K_{n+2}+K_n, \qquad K_0=0, K_1=1, K_2=-1, &(2.2)\\ S_{n+3} &= -S_{n+2}+S_n, \qquad S_0=3, S_1=-1, S_2=1, &(2.3)\\ R_{n+3} &= -R_{n+2}+R_n, \qquad R_0=1, R_1=0, R_2=-1, &(2.4)\end{aligned}$$

i.e,

$$\begin{aligned} K_n &= -K_{n-1}+K_{n-3}, \qquad K_0=0, K_1=1, K_2=-1,\\ S_n &= -S_{n-1}+S_{n-3}, \qquad S_0=3, S_1=-1, S_2=1,\\ R_n &= -R_{n-1}+R_{n-3}, \qquad R_0=1, R_1=0, R_2=-1.\end{aligned}$$

The sequences $\{R_n\}_{n\geq 0}$, $\{K_n\}_{n\geq 0}$ and $\{S_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} K_{-n} &= K_{-(n-2)}+K_{-(n-3)},\\ S_{-n} &= S_{-(n-2)}+S_{-(n-3)},\\ R_{-n} &= R_{-(n-2)}+R_{-(n-3)}.\end{aligned}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (2.2)-(2.4) hold for all integers n.

Next, we present the first few values of the co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers with positive and negative subscripts.

Table 2. The first few values of the special third-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
K_n	0	1	−1	1	0	−1	2	−2	1	1	−3	4	−3	0
K_{-n}	0	0	1	0	1	1	1	2	2	3	4	5	7	9
S_n	3	−1	1	2	−3	4	−2	−1	5	−7	6	−1	−6	12
S_{-n}	3	0	2	3	2	5	5	7	10	12	17	22	29	39
R_n	1	0	−1	2	−2	1	1	−3	4	−3	0	4	−7	7
R_{-n}	1	−1	1	0	0	1	0	1	1	1	2	2	3	4

For all integers n, Binet's formula of co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers (using initial conditions (2.2)-(2.4) in Theorem 2.2) can be expressed as follows:

Theorem 2.3. *For all integers n, Binet's formulas of co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers are*

$$\begin{aligned} K_n &= \frac{\theta_1^{n+1}}{(\theta_1-\theta_2)(\theta_1-\theta_3)}+\frac{\theta_2^{n+1}}{(\theta_2-\theta_1)(\theta_2-\theta_3)}+\frac{\theta_3^{n+1}}{(\theta_3-\theta_1)(\theta_3-\theta_2)}\\ &= \frac{\theta_1^{n+2}}{-\theta_1^2+3}+\frac{\theta_2^{n+2}}{-\theta_2^2+3}+\frac{\theta_3^{n+2}}{-\theta_3^2+3},\end{aligned}$$

and

$$S_n = \theta_1^n + \theta_2^n + \theta_3^n,$$

and

$$R_n = \frac{\theta_1^{n+4}}{(\theta_1 - \theta_2)(\theta_1 - \theta_3)} + \frac{\theta_2^{n+4}}{(\theta_2 - \theta_1)(\theta_2 - \theta_3)} + \frac{\theta_3^{n+4}}{(\theta_3 - \theta_1)(\theta_3 - \theta_2)}$$

respectively.

Lemma 2.2.1 gives the following results as particular examples (of generating functions of co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers).

Corollary 2.4. *Generating functions of co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers are*

$$\begin{aligned} \sum_{n=0}^{\infty} K_n z^n &= \frac{z}{1+z-z^3}, \\ \sum_{n=0}^{\infty} S_n z^n &= \frac{3+2z}{1+z-z^3}, \\ \sum_{n=0}^{\infty} R_n z^n &= \frac{1+z-z^2}{1+z-z^3}, \end{aligned}$$

respectively.

3. Connections between U_n, E_n and K_n, S_n

S_n can be given as follows.

Lemma 3.0.1. *For all integers n, we have the following formula for S_n:*

$$\begin{aligned} S_n &= \theta_1^n + \theta_2^n + \theta_3^n \\ &= \alpha^n \beta^n + \alpha^n \gamma^n + \beta^n \gamma^n. \end{aligned}$$

Proof. Use [4, Lemma 30.]. □

We can present the relations between K_n, S_n and U_n, E_n as follows.

Lemma 3.0.2. *For all integers n, we have the following formulas:*

(a) $S_n = \frac{1}{2}(E_n^2 - E_{2n})$.
(b) $K_n = U_{-n-1}$ *and* $K_{-n} = U_{n-1}$.
(c) $S_n = E_{-n}$ *and* $S_{-n} = E_n$.

Proof. Use [4, Lemma 32.]. □

4. Some Identities of Generalized co-Narayana Numbers

In this section, we obtain some identities of co-adjusted Padovan, co-Perrin (co-Padovan-Lucas) and co-Padovan numbers. First, we can give a few basic relations between $\{K_n\}$ and $\{S_n\}$.

Lemma 4.0.1. *The following equalities are true:*

(a) $S_n = 3K_{n+4} + 5K_{n+3} + 2K_{n+2}$.
(b) $S_n = 2K_{n+3} + 2K_{n+2} + 3K_{n+1}$.
(c) $S_n = 3K_{n+1} + 2K_n$.
(d) $S_n = 3K_{n+1} + 2K_n$.
(e) $S_n = -K_n + 3K_{n-2}$.
(f) $23K_n = 3S_{n+4} + S_{n+3} + 7S_{n+2}$.
(g) $23K_n = -2S_{n+3} + 7S_{n+2} + 3S_{n+1}$.
(h) $23K_n = 9S_{n+2} + 3S_{n+1} - 2S_n$.
(i) $23K_n = -6S_{n+1} - 2S_n + 9S_{n-1}$.
(j) $23K_n = 4S_n + 9S_{n-1} - 6S_{n-2}$.

Proof. Set $U_n = K_n$, $S_n = S_n$ and $r = -1$, $s = 0$, $t = 1$ in [4, Lemma 36.]. □
Note that all the identities in the above lemma can be proved by induction as well.
Next, we give a few basic relations between $\{K_n\}$ and $\{Y_n\}$.

Lemma 4.0.2. *The following equalities are true:*

(a) $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + 2Y_1Y_2^2 - Y_0^2Y_2 - Y_0Y_1^2 - 3Y_0Y_1Y_2)K_n = (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+2} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n$.
(b) $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + 2Y_1Y_2^2 - Y_0^2Y_2 - Y_0Y_1^2 - 3Y_0Y_1Y_2)K_n = (Y_2^2 + Y_1^2 - Y_0^2 + 2Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n + (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n-1}$.
(c) $(Y_2^3+Y_1^3+Y_0^3+Y_1^2Y_2+2Y_1Y_2^2-Y_0^2Y_2-Y_0Y_1^2-3Y_0Y_1Y_2)K_n = (-Y_2^2+Y_0^2-2Y_1Y_2-Y_0Y_2+Y_0Y_1)Y_n + (-Y_1^2+Y_0^2-Y_1Y_2)Y_{n-1} + (Y_2^2+Y_1^2-Y_0^2+2Y_1Y_2-Y_0Y_1)Y_{n-2}$.
(d) $Y_n = (Y_2 + Y_1)K_{n+2} + (Y_2 + Y_1 + Y_0)K_{n+1} + (Y_1 + Y_0)K_n$.
(e) $Y_n = Y_0K_{n+1} + (Y_1 + Y_0)K_n + (Y_2 + Y_1)K_{n-1}$.
(f) $Y_n = Y_1K_n + (Y_2 + Y_1)K_{n-1} + Y_0K_{n-2}$.

Proof. Set $W_n = Y_n$, $U_n = K_n$ and $r = -1$, $s = 0$, $t = 1$ in [4, Lemma 37.]. □
Now, we present a few basic relations between $\{S_n\}$ and $\{Y_n\}$.

Lemma 4.0.3. *The following equalities are true:*

(a) $(Y_2^3+Y_1^3+Y_0^3+Y_1^2Y_2+2Y_1Y_2^2-Y_0^2Y_2-Y_0Y_1^2-3Y_0Y_1Y_2)S_n = (3Y_2^2+Y_1^2-Y_0^2+4Y_1Y_2-3Y_0Y_1)Y_{n+2} + (2Y_2^2+3Y_1^2-3Y_0Y_2+2Y_1Y_2-2Y_0Y_1)Y_{n+1} + (-Y_1^2+3Y_0^2-3Y_1Y_2-2Y_0Y_2)Y_n$.
(b) $(Y_2^3 + Y_1^3 + Y_0^3 + Y_1^2Y_2 + 2Y_1Y_2^2 - Y_0^2Y_2 - Y_0Y_1^2 - 3Y_0Y_1Y_2)S_n = (-Y_1^2 - Y_2^2 + 3Y_1^2 + Y_0^2 - 3Y_0Y_2 - 2Y_1Y_2 + Y_0Y_1)Y_{n+1} + (3Y_0^2 - 3Y_1Y_2 - Y_1^2 - 2Y_0Y_2)Y_n + (3Y_2^2 + Y_1^2 - Y_0^2 + 4Y_1Y_2 - 3Y_0Y_1)Y_{n-1}$.
(c) $(Y_2^3+Y_1^3+Y_0^3+Y_1^2Y_2+2Y_1Y_2^2-Y_0^2Y_2-Y_0Y_1^2-3Y_0Y_1Y_2)S_n = (Y_2^2-3Y_1^2+2Y_0^2-Y_1Y_2+Y_0Y_2-Y_0Y_1)Y_n + (3Y_2^2+Y_1^2-Y_0^2+4Y_1Y_2-3Y_0Y_1)Y_{n-1} + (-Y_2^2+2Y_1^2+Y_0^2-2Y_1Y_2-3Y_0Y_2+Y_0Y_1)Y_{n-2}$.
(d) $23Y_n = (-2Y_2 + 7Y_1 + 3Y_0)S_{n+2} + (7Y_2 + 10Y_1 + Y_0)S_{n+1} + (3Y_2 + Y_1 + 7Y_0)S_n$.
(e) $23Y_n = (9Y_2 + 3Y_1 - 2Y_0)S_{n+1} + (3Y_2 + Y_1 + 7Y_0)S_n + (-2Y_2 + 7Y_1 + 3Y_0)S_{n-1}$.
(f) $23Y_n = (-6Y_2 - 2Y_1 + 9Y_0)S_n + (-2Y_2 + 7Y_1 + 3Y_0)S_{n-1} + (9Y_2 + 3Y_1 - 2Y_0)S_{n-2}$.

Proof. Set $W_n = Y_n$, $S_n = S_n$, and $r = -1$, $s = 0$, $t = 1$ in [4, Lemma 38.]. $\square$

Next, we give a few basic relations between $\{K_n\}$ and $\{R_n\}$.

Lemma 4.0.4. *The following equalities are true:*

(a) $K_n = R_{n+2} + R_{n+1} + R_n$.
(b) $K_n = R_n + R_{n-1}$.
(c) $K_n = R_n + R_{n-1}$.
(d) $R_n = -K_{n+2} + K_n$.
(e) $R_n = K_{n+1} + K_n - K_{n-1}$.
(f) $R_n = -K_{n-1} + K_{n-2}$.

Proof. Set $Y_n = R_n$ in Lemma 4.0.2. $\square$

Now, we present a few basic relations between $\{S_n\}$ and $\{R_n\}$.

Lemma 4.0.5. *The following equalities are true:*

(a) $S_n = 2R_{n+2} + 5R_{n+1} + 5R_n$.
(b) $S_n = 3R_{n+1} + 5R_n + 2R_{n-1}$.
(c) $S_n = 2R_n + 2R_{n-1} + 3R_{n-2}$.
(d) $23R_n = 5S_{n+2} - 6S_{n+1} + 4S_n$.
(e) $23R_n = -11S_{n+1} + 4S_n + 5S_{n-1}$.
(f) $23R_n = 15S_n + 5S_{n-1} - 11S_{n-2}$.

Proof. Set $Y_n = R_n$ in Lemma 4.0.3. $\square$

We can present identities between P_n, E_n, U_n and R_n, K_n, S_n by using Lemmas given above.

Lemma 4.0.6. *For all integers n, we have the following formulas:*

(a) $S_{-n} = -U_{n+2} + 3U_{n+1} + U_n$.
(b) $S_{-n} = -3P_{n+2} + 2P_{n+1} + 4P_n$.
(c) $K_{-n} = P_{n+1} - P_n$.
(d) $23K_{-n} = -6E_{n+2} + 9E_{n+1} + 4E_n$.
(e) $S_{-n} = 3U_{n-2} + 2U_{n-1}$.
(f) $23K_{-n} = 9E_{n-2} + 3E_{n-1} - 2E_n$.
(g) $R_{-n} = -2P_{n+2} + 2P_{n+1} + P_n$.
(h) $U_{-n-1} = R_{n+2} + R_{n+1} + R_n$.
(i) $R_{-n} = U_{n+1} - U_n$.
(j) $23R_{-n} = -11E_{n+2} + 5E_{n+1} + 15E_n$.
(k) $S_n = \frac{1}{2}((-U_{n+2} + 3U_{n+1} + U_n)^2 + (U_{2n+2} - 3U_{2n+1} - U_{2n}))$.

Prof. Use Lemmas 3.0.2, 1.3.1, 1.3.2, 1.3.3, 4.0.1, 4.0.4, 4.0.5. $\square$

Now, we present some identities of generalized co-Tribonacci numbers and its special cases.

Lemma 4.0.7. *Suppose that $\{X_n\}_{n\geq 0} = \{X_n(X_0, X_1, X_2)\}_{n\geq 0}$ is also defined by the third-order recurrence relations*

$$X_n = -X_{n-1} + X_{n-3} \tag{4.1}$$

i.e.,

$$X_{n+3} = -X_{n+2} + X_n$$

with the initial values X_0, X_1, X_2 *not all being zero and*

$$X_{-n} = X_{-(n-2)} + X_{-(n-3)}$$

so that (4.1) is true for all integer n.

Then the following equalities are true:

(a)

$$(X_0X_3^2 + X_1^2X_4 + X_2^3 - X_0X_2X_4 - 2X_1X_2X_3)Y_n = q_1X_{n+2} + q_2X_{n+1} + q_3X_n$$

where

$q_1 = (X_1^2 - X_0X_2)Y_2 + (X_0X_3 - X_1X_2)Y_1 + (X_2^2 - X_1X_3)Y_0,$
$q_2 = (X_0X_3 - X_1X_2)Y_2 + (X_2^2 - X_0X_4)Y_1 + (X_1X_4 - X_2X_3)Y_0,$
$q_3 = (X_2^2 - X_1X_3)Y_2 + (X_1X_4 - X_2X_3)Y_1 + (X_3^2 - X_2X_4)Y_0.$

(b)

$$(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)K_n = q_4Y_{n+2} + q_5Y_{n+1} + q_6Y_n$$

where

$q_4 = -Y_1^2 + Y_0^2 - Y_1Y_2,$
$q_5 = Y_2^2 + Y_1Y_2 - Y_0Y_1,$
$q_6 = Y_1^2 - Y_0Y_2.$

(c)

$$Y_n = q_7K_{n+2} + q_8K_{n+1} + q_9K_n$$

where

$q_7 = Y_2 + Y_1,$
$q_8 = Y_2 + Y_1 + Y_0,$
$q_9 = Y_1 + Y_0.$

(d)

$$(Y_0Y_3^2 + Y_1^2Y_4 + Y_2^3 - Y_0Y_2Y_4 - 2Y_1Y_2Y_3)S_n = q_{10}Y_{n+2} + q_{11}Y_{n+1} + q_{12}Y_n$$

where

$q_{10} = 3Y_2^2 + Y_1^2 - Y_0^2 + 4Y_1Y_2 - 3Y_0Y_1,$
$q_{11} = 2Y_2^2 + 3Y_1^2 + 2Y_1Y_2 - 3Y_0Y_2 - 2Y_0Y_1,$
$q_{12} = -Y_1^2 + 3Y_0^2 - 3Y_1Y_2 - 2Y_0Y_2.$

(e)

$$23Y_n = q_{13}S_{n+2} + q_{14}S_{n+1} + q_{15}S_n$$

where

$q_{13} = -2Y_2 + 7Y_1 + 3Y_0,$
$q_{14} = 7Y_2 + 10Y_1 + Y_0,$

$q_{15} = 3Y_2 + (Y_1 + 7Y_0.$

Proof.

(a) Writing

$$Y_n = q_1 \times X_{n+2} + q_2 \times X_{n+1} + q_3 \times X_n$$

and solving the system of equations

$$\begin{array}{rcl} Y_0 & = & q_1 \times X_2 + q_2 \times X_1 + q_3 \times X_0 \\ Y_1 & = & q_1 \times X_3 + q_2 \times X_2 + q_3 \times X_1 \\ Y_2 & = & q_1 \times X_4 + q_2 \times X_3 + q_3 \times X_2 \end{array}$$

we find the required identity.

(b) Replace Y_n and X_n with K_n and Y_n, respectively in (a).

(c) Replace X_n with K_n in (a).

(d) Replace Y_n and X_n with S_n and Y_n, respectively in (a).

(e) Replace X_n with S_n in (a). □

5. Simson's Formulas of co-Padovan Numbers

The following theorem gives Simson's formula of the generalized co-Padovan numbers $\{Y_n\}$.

Theorem 5.1 (Simson's Formula of Generalized co-Padovan Numbers)**.** *For all integers n, we have*

$$\begin{aligned} \begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} &= \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix} \\ &= \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_2 + Y_1 \\ Y_0 & Y_2 + Y_1 & Y_1 + Y_0 \end{vmatrix} \end{aligned}$$

Proof. Set $W_n = Y_n$ and $r = -1$, $s = 0$, $t = 1$ in [4, Theorem 33.]. □

The previous theorem gives the following results as particular examples.

Corollary 5.2. *For all integers n, co-adjusted Padovan, co-Perrin (co-Padovan-*

Lucas) and co-Padovan are given as

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = -1,$$

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{vmatrix} = -23,$$

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_n \\ R_{n+1} & R_n & R_{n-1} \\ R_n & R_{n-1} & R_{n-2} \end{vmatrix} = -1,$$

respectively.

Proof. Set $Y_n = R_n$, $Y_n = K_n$ and $Y_n = S_n$ in Theorem 5.1, respectively. □

6. Recurrence Properties of Generalized co-Padovan Numbers

The generalized co-Tribonacci numbers W_n at negative indices can be expressed by the sequence itself at positive indices.

Theorem 6.1. *For $n \in \mathbb{Z}$, we have*

$$Y_{-n} = Y_{2n} - S_n Y_n + \frac{1}{2}(S_n^2 - S_{2n})Y_0.$$

Proof. Set $W_n = Y_n$, $H_n = S_n$ and $r = -1$, $s = 0$, $t = 1$ in [4, Theorem 39.]. □
As special cases of the above Theorem, we have the following Corollary.

Corollary 6.2. *For $n \in \mathbb{Z}$, we have*

(a) $K_{-n} = -2K_n^2 + K_{2n} - 3K_{n+1}K_n$.

(b) $S_{-n} = \dfrac{1}{2}(S_n^2 - S_{2n})$.

(c) $R_{-n} = R_{2n} - S_n R_n + \frac{1}{2}(S_n^2 - S_{2n})$.
$= \frac{1}{2}(4R_{n+2}^2 + 25R_{n+1}^2 + 15R_n^2 - 2R_{2n+2} - 5R_{2n+1} - 3R_{2n} + 20R_{n+1}R_{n+2} + 16R_nR_{n+2} + 40R_nR_{n+1})$.

Proof. For (a) and (b) take $r = -1$, $s = 0$, $t = 1$, and $G_n = K_n$ and $H_n = S_n$,respectively, in [9.2, Corollary 42.] or set $Y_n = K_n$ and $Y_n = S_n$, respectively, in Theorem 6.1. For (c) take $Y_n = R_n$ in Theorem 6.1 and use the identity $S_n = 2R_{n+2} + 5R_{n+1} + 5R_n$ which is given in Lemma 4.0.5. □
The last Corollary can be written in the following form by using Lemma 3.0.2.

Corollary 6.3. *For $n \in \mathbb{Z}$, we have*

(a) $U_{n-1} = -2K_n^2 + K_{2n} - 3K_{n+1}K_n$.

(b) $E_n = \dfrac{1}{2}(S_n^2 - S_{2n})$.

Proof. Use Lemma 3.0.2 and Corollary 6.2. □

7. Sum Formulas $\sum\limits_{k=0}^{n} Y_k$, $\sum\limits_{k=0}^{n} Y_{2k}$, $\sum\limits_{k=0}^{n} Y_{2k+1}$, $\sum\limits_{k=0}^{n} Y_{-k}$, $\sum\limits_{k=0}^{n} Y_{-2k}$, $\sum\limits_{k=0}^{n} Y_{-2k+1}$ and Generating Functions $\sum_{n=0}^{\infty} Y_n z^n$, $\sum_{n=0}^{\infty} Y_{2n} z^n$, $\sum_{n=0}^{\infty} Y_{2n+1} z^n$, $\sum_{n=0}^{\infty} Y_{-n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n} z^n$, $\sum_{n=0}^{\infty} Y_{-2n+1} z^n$ of Generalized co-Padovan Numbers

Next, we present sum formulas of generalized co-Padovan numbers

Theorem 7.1. *For $n \geq 0$, we have the following sum formulas for generalized co-Padovan numbers:*

(a) $\sum\limits_{k=0}^{n} Y_k = -Y_{n+2} - 2Y_{n+1} - Y_n + Y_2 + 2Y_1 + 2Y_0.$

(b) $\sum\limits_{k=0}^{n} Y_{2k} = -Y_{2n+2} - Y_{2n+1} + Y_2 + Y_1 + Y_0.$

(c) $\sum\limits_{k=0}^{n} Y_{2k+1} = -Y_{2n} + Y_1 + Y_0.$

(d) $\sum\limits_{k=0}^{n} Y_{-k} = Y_{-n+2} + 2Y_{-n+1} + 2Y_{-n} - Y_2 - 2Y_1 - Y_0.$

(e) $\sum\limits_{k=0}^{n} Y_{-2k} = Y_{-2n-1} + Y_{-2n} - Y_2 - Y_1.$

(f) $\sum\limits_{k=0}^{n} Y_{-2k+1} = Y_{-2n-2} - Y_0.$

Proof.

(a) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (a) (i)].
(b) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (b) (i)].
(c) Set $W_n = Y_n, r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (c) (i)].
(d) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (d) (i)].
(e) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (e) (i)].
(f) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [4, Theorem 62 (f) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-adjusted Padovan numbers (take $Y_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = -1$).

Corollary 7.2. *For $n \geq 0$, co-adjusted Padovan numbers have the following properties.*

(a) $\sum\limits_{k=0}^{n} K_k = -K_{n+2} - 2K_{n+1} - K_n + 1.$

(b) $\sum\limits_{k=0}^{n} K_{2k} = -K_{2n+2} - K_{2n+1}.$

(c) $\sum\limits_{k=0}^{n} K_{2k+1} = -K_{2n} + 1.$

(d) $\sum\limits_{k=0}^{n} K_{-k} = K_{-n+2} + 2K_{-n+1} + 2K_{-n} - 1.$

(e) $\sum\limits_{k=0}^{n} K_{-2k} = K_{-2n-1} + K_{-2n}.$

(f) $\sum\limits_{k=0}^{n} K_{-2k+1} = K_{-2n-2}.$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Perrin (co-Padovan-Lucas) numbers.

Corollary 7.3. *For $n \geq 0$, co-Perrin (co-Padovan-Lucas) numbers have the following properties:*

(a) $\sum_{k=0}^{n} S_k = -S_{n+2} - 2S_{n+1} - S_n + 5.$

(b) $\sum_{k=0}^{n} S_{2k} = -S_{2n+2} - S_{2n+1} + 3.$

(c) $\sum_{k=0}^{n} S_{2k+1} = -S_{2n} + 2.$

(d) $\sum_{k=0}^{n} S_{-k} = S_{-n+2} + 2S_{-n+1} + 2S_{-n} - 2.$

(e) $\sum_{k=0}^{n} S_{-2k} = S_{-2n-1} + S_{-2n}.$

(f) $\sum_{k=0}^{n} S_{-2k+1} = S_{-2n-2} - 3.$

From the last Theorem, we have the following Corollary which gives sum formulas of co-Padovan numbers (take $Y_n = R_n$ with $R_0 = 1, R_1 = 0, R_2 = -1$).

Corollary 7.4. *For $n \geq 0$, co-Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} R_k = -R_{n+2} - 2R_{n+1} - R_n + 1.$

(b) $\sum_{k=0}^{n} R_{2k} = -R_{2n+2} - R_{2n+1}.$

(c) $\sum_{k=0}^{n} R_{2k+1} = -R_{2n} + R_1 + 1.$

(d) $\sum_{k=0}^{n} R_{-k} = R_{-n+2} + 2R_{-n+1} + 2R_{-n}.$

(e) $\sum_{k=0}^{n} R_{-2k} = R_{-2n-1} + R_{-2n} + 1.$

(f) $\sum_{k=0}^{n} R_{-2k+1} = R_{-2n-2} - 1.$

Next, we give the ordinary generating function of special cases of the generalized co-Padovan numbers $\{Y_{mn+j}\}$.

Corollary 7.5. *The ordinary generating functions of the sequences Y_n, Y_{2n}, Y_{2n+1}, Y_{-n}, Y_{-2n}, Y_{-2n+1} are given as follows:*

(a) $(|z| < \min\{|\theta_1|^{-1}, |\theta_2|^{-1}, |\theta_3|^{-1}\} = |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.868836).$

$$\sum_{n=0}^{\infty} Y_n z^n = \frac{(Y_2 + Y_1)z^2 + (Y_1 + Y_0)z + Y_0}{-z^3 + z + 1}$$

(b) $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2}\} \simeq 0.754877)$.

$$\sum_{n=0}^{\infty} Y_{2n}z^n = \frac{(Y_1+Y_0)z^2+(Y_2-Y_0)z+Y_0}{-z^3+2z^2-z+1}$$

(c) $(|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}\} = |\theta_2|^{-2} = |\theta_3|^{-2}\} \simeq 0.754877)$.

$$\sum_{n=0}^{\infty} Y_{2n+1}z^n = \frac{(Y_2+Y_1)z^2+(-Y_2-Y_1+Y_0)z+Y_1}{-z^3+2z^2-z+1}$$

(d) $(|z| < \min\{|\theta_1|, |\theta_2|, |\theta_3|\} = |\theta_1| \simeq 0.754877)$.

$$\sum_{n=0}^{\infty} Y_{-n}z^n = \frac{Y_1z^2+(Y_2+Y_1)z+Y_0}{-z^3-z^2+1}$$

(e) $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.569840)$.

$$\sum_{n=0}^{\infty} Y_{-2n}z^n = \frac{Y_2z^2+(Y_1-Y_0)z+Y_0}{-z^3+z^2-2z+1}$$

(f) $(|z| < \min\{|\theta_1|^2, |\theta_2|^2, |\theta_3|^2\} = |\theta_1|^2 \simeq 0.569840)$.

$$\sum_{n=0}^{\infty} Y_{-2n+1}z^n = \frac{(-Y_2+Y_0)z^2+(Y_2-Y_1)z+Y_1}{-z^3+z^2-2z+1}$$

Proof. Set $W_n = Y_n$ and $r = -1$, $s = 0$, $t = 1$ in [4, Corollary 67.]. □
Now, we consider special cases of the last corollary.

Corollary 7.6. *The ordinary generating functions of special cases of the generalized co-Padocan numbers are given as follows:*

(a) $(|z| < |\theta_2|^{-1} = |\theta_3|^{-1} \simeq 0.868836)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_nz^n &= \frac{z}{-z^3+z+1}\\ \sum_{n=0}^{\infty} S_nz^n &= \frac{2z+3}{-z^3+z+1}\\ \sum_{n=0}^{\infty} R_nz^n &= \frac{-z^2+z+1}{-z^3+z+1}\end{aligned}$$

(b) $(|z| < |\theta_2|^{-2} = |\theta_3|^{-2}\} \simeq 0.754877)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_{2n}z^n &= \frac{z^2-z}{-z^3+2z^2-z+1}\\ \sum_{n=0}^{\infty} S_{2n}z^n &= \frac{2z^2-2z+3}{-z^3+2z^2-z+1}\\ \sum_{n=0}^{\infty} R_{2n}z^n &= \frac{z^2-2z+1}{-z^3+2z^2-z+1}\end{aligned}$$

(c) $(|z| < |\theta_2|^{-2} = |\theta_3|^{-2}\} \simeq 0.754877)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_{2n+1}z^n &= \frac{1}{-z^3+2z^2-z+1}\\ \sum_{n=0}^{\infty} S_{2n+1}z^n &= \frac{3z-1}{-z^3+2z^2-z+1}\\ \sum_{n=0}^{\infty} R_{2n+1}z^n &= \frac{2z-z^2}{-z^3+2z^2-z+1}\end{aligned}$$

(d) $(|z| < |\theta_1| \simeq 0.754877)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_{-n}z^n &= \frac{-z^2}{z^3+z^2-1}\\ \sum_{n=0}^{\infty} S_{-n}z^n &= \frac{z^2-3}{z^3+z^2-1}\\ \sum_{n=0}^{\infty} R_{-n}z^n &= \frac{z-1}{z^3+z^2-1}\end{aligned}$$

(e) $(|z| < |\theta_1|^2 \simeq 0.569840)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_{-2n}z^n &= \frac{z^2-z}{z^3-z^2+2z-1}\\ \sum_{n=0}^{\infty} S_{-2n}z^n &= \frac{-z^2+4z-3}{z^3-z^2+2z-1}\\ \sum_{n=0}^{\infty} R_{-2n}z^n &= \frac{z^2+z-1}{z^3-z^2+2z-1}\end{aligned}$$

(f) $(|z| < |\theta_1|^2 \simeq 0.569840)$.

$$\begin{aligned}\sum_{n=0}^{\infty} K_{-2n+1}z^n &= \frac{-z^2+2z-1}{z^3-z^2+2z-1}\\ \sum_{n=0}^{\infty} S_{-2n+1}z^n &= \frac{-2z^2-2z+1}{z^3-z^2+2z-1}\\ \sum_{n=0}^{\infty} R_{-2n+1}z^n &= \frac{-2z^2+z}{z^3-z^2+2z-1}\end{aligned}$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 7.7. *We have the following infinite sums .*

(a) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{K_n}{2^n} &= \frac{4}{11},\\ \sum_{n=0}^{\infty} \frac{S_n}{2^n} &= \frac{32}{11},\\ \sum_{n=0}^{\infty} \frac{R_n}{2^n} &= \frac{10}{11}.\end{aligned}$$

(b) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{K_{2n}}{2^n} &= -\frac{2}{7},\\ \sum_{n=0}^{\infty} \frac{S_{2n}}{2^n} &= \frac{20}{7},\\ \sum_{n=0}^{\infty} \frac{R_{2n}}{2^n} &= \frac{2}{7}.\end{aligned}$$

(c) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{K_{2n+1}}{2^n} &= \frac{8}{7},\\ \sum_{n=0}^{\infty} \frac{S_{2n+1}}{2^n} &= \frac{4}{7},\\ \sum_{n=0}^{\infty} \frac{R_{2n+1}}{2^n} &= \frac{6}{7}.\end{aligned}$$

(d) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{K_{-n}}{2^n} &= \frac{2}{5},\\ \sum_{n=0}^{\infty}\frac{S_{-n}}{2^n} &= \frac{22}{5},\\ \sum_{n=0}^{\infty}\frac{R_{-n}}{2^n} &= \frac{4}{5}.\end{aligned}$$

(e) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{K_{-2n}}{2^n} &= 2,\\ \sum_{n=0}^{\infty}\frac{S_{-2n}}{2^n} &= 10,\\ \sum_{n=0}^{\infty}\frac{R_{-2n}}{2^n} &= 2.\end{aligned}$$

(f) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{K_{-2n+1}}{2^n} &= 2,\\ \sum_{n=0}^{\infty}\frac{S_{-2n+1}}{2^n} &= 4,\\ \sum_{n=0}^{\infty}\frac{R_{-2n+1}}{2^n} &= 0.\end{aligned}$$

8. Sum Formulas $\sum_{k=0}^{n} z^k Y_k^2$, $\sum_{k=0}^{n} z^k Y_{k+1}Y_k$, $\sum_{k=0}^{n} z^k Y_{k+2}Y_k$ and Generating Functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of Generalized co-padovan Numbers Numbers

Next, we present sum formulas of generalized co-Padovan Numbers numbers.

Theorem 8.1. *For $n \geq 0$, we have the following sum formulas for generalized co-Padovan Numbers numbers:*

(a) $\sum\limits_{k=0}^{n} Y_k^2 = Y_{n+3}^2 - 2Y_{n+1}Y_{n+3} - 2Y_{n+1}Y_{n+2} - Y_2^2 + 2Y_0Y_2 + 2Y_0Y_1.$

(b) $\sum\limits_{k=0}^{n} Y_{k+1}Y_k = -Y_{n+3}^2 - Y_{n+2}^2 - Y_{n+1}^2 - Y_{n+2}Y_{n+3} + Y_{n+1}Y_{n+3} + Y_2^2 + Y_1^2 + Y_0^2 + Y_1Y_2 - Y_0Y_2.$

(c) $\sum\limits_{k=0}^{n} Y_{k+2}Y_k = -Y_{n+1}Y_{n+3} - Y_{n+1}Y_{n+2} + Y_0Y_2 + Y_0Y_1.$

Proof.

Note that characteristic equation of the third-order recurrence sequence Y_n is the cubic equation $y^3 + y^2 - 1 = 0$ whose roots are $\theta_1, \theta_2, \theta_3$ with $\theta_1 \neq \theta_2 \neq \theta_3$. In [5, Theorem 2.1]), for $r = -1$, $s = 0$, $t =$, we get

$$\begin{aligned} \Gamma(z) &= (-t^2z^3 + sz + rtz^2 + 1)(r^2z - s^2z^2 + t^2z^3 + 2sz + 2rtz^2 - 1) \\ &= (z^3 + z^2 - 1)(-z^3 + 2z^2 - z + 1) \\ &= -z^6 + z^5 + z^4 + z^3 - z^2 + z - 1 \end{aligned}$$

and $\Gamma(1) \neq 0$.

(a) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [5, Theorem 2.1 (a) (i)] or in [6 , Theorem 2.1 (a) (i)].
(b) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [5, Theorem 2.1 (b) (i)] or in [6 , Theorem 2.1 (b) (i)].
(c) Set $W_n = Y_n$, $r = -1$, $s = 0$, $t = 1$ and $z = 1$ in [5, Theorem 2.1 (c) (i)] or in [6 , Theorem 2.1 (c) (i)]. □

From the last Theorem, we have the following Corollary which gives sum formulas of co-adjusted Padovan numbers (take $Y_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = -1$).

Corollary 8.2. *For $n \geq 0$, co-adjusted Padovan numbers have the following properties.*

(a) $\sum\limits_{k=0}^{n} K_k^2 = K_{n+3}^2 - 2K_{n+1}K_{n+3} - 2K_{n+1}K_{n+2} - 1.$
(b) $\sum\limits_{k=0}^{n} K_{k+1}K_k = -K_{n+3}^2 - K_{n+2}^2 - K_{n+1}^2 - K_{n+2}K_{n+3} + K_{n+1}K_{n+3} + 1.$
(c) $\sum\limits_{k=0}^{n} K_{k+2}K_k = -K_{n+1}K_{n+3} - K_{n+1}K_{n+2}.$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 = 1$ in the last Theorem, we have the following Corollary which gives sum formulas of co-Perrin (co-Padovan-Lucas) numbers.

Corollary 8.3. *For $n \geq 0$, co-Perrin (co-Padovan-Lucas) numbers have the following properties:*

(a) $\sum\limits_{k=0}^{n} S_k^2 = S_{n+3}^2 - 2S_{n+1}S_{n+3} - 2S_{n+1}S_{n+2} - 1.$
(b) $\sum\limits_{k=0}^{n} S_{k+1}S_k = -S_{n+3}^2 - S_{n+2}^2 - S_{n+1}^2 - S_{n+2}S_{n+3} + S_{n+1}S_{n+3} + 7.$
(c) $\sum\limits_{k=0}^{n} S_{k+2}S_k = -S_{n+1}S_{n+3} - S_{n+1}S_{n+2}.$

From the last Theorem, we have the following Corollary which gives sum formulas of co-Padovan numbers (take $Y_n = R_n$ with $R_0 = 1, R_1 = 0, R_2 = -1$).

Corollary 8.4. *For $n \geq 0$, co-Padovan numbers have the following properties.*

(a) $\sum_{k=0}^{n} R_k^2 = R_{n+3}^2 - 2R_{n+1}R_{n+3} - 2R_{n+1}R_{n+2} - 3.$

(b) $\sum_{k=0}^{n} R_{k+1}R_k = -R_{n+3}^2 - R_{n+2}^2 - R_{n+1}^2 - R_{n+2}R_{n+3} + R_{n+1}R_{n+3} + 3.$

(c) $\sum_{k=0}^{n} R_{k+2}R_k = -R_{n+1}R_{n+3} - R_{n+1}R_{n+2} - 1.$

Next, we give the ordinary generating functions $\sum_{n=0}^{\infty} Y_n^2 z^n$, $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n$, $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n$ of the sequences $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$.

Theorem 8.5. *Assume that* $|z| < \min\{|\theta_1|^{-2}, |\theta_2|^{-2}, |\theta_3|^{-2}, |\theta_1\theta_2|^{-1}, |\theta_1\theta_3|^{-1}, |\theta_2\theta_3|^{-1}\} = |\theta_2\theta_3|^{-1} = |\theta_3|^{-2} = |\theta_2|^{-2} \simeq 0.754877$. *Then the ordinary generating functions of the sequences* $\{Y_n^2\}$, $\{Y_{n+1}Y_n\}$, $\{Y_{n+2}Y_n\}$ *are given as follows:*

(a) $\sum_{n=0}^{\infty} Y_n^2 z^n = \frac{1}{-z^6+z^5+z^4+z^3-z^2+z-1}(z^5(Y_1+Y_2)^2 + z^4(Y_0^2+2Y_1Y_0-Y_2^2-2Y_1Y_2) + z^3(2Y_0Y_2-Y_1^2) - z^2(Y_2^2-Y_1^2+Y_0^2) + z(Y_0^2-Y_1^2) - Y_0^2).$

(b) $\sum_{n=0}^{\infty} Y_{n+1}Y_n z^n = \frac{1}{-z^6+z^5+z^4+z^3-z^2+z-1}(z^5Y_0(Y_1+Y_2) + z^4Y_1(Y_1+Y_2) + z^3Y_0(Y_0-Y_2) - z^2(Y_1+Y_2)(Y_0-Y_2) + zY_1(Y_0-Y_2) - Y_0Y_1).$

(c) $\sum_{n=0}^{\infty} Y_{n+2}Y_n z^n = \frac{1}{-z^6+z^5+z^4+z^3-z^2+z-1}(z^5Y_1(Y_1+Y_2) + z^4(Y_0^2-Y_1^2+Y_0Y_1-Y_1Y_2) + z^3(Y_2^2-Y_0^2+Y_1Y_2+Y_0Y_2) + z^2(-Y_2^2-2Y_1Y_2+Y_0Y_1) + z(Y_1Y_2+Y_0Y_2-Y_0Y_1) - Y_0Y_2).$

Proof. Set $W_n = Y_n$ and $r = -1$, $s = 0$, $t = 1$ in [5 , Theorem 3.1] or in [6, Theorem 3.1]. □

Now, we consider special cases of the last Theorem.

Corollary 8.6. *Assume that* $|z| < |\theta_2\theta_3|^{-1} = |\theta_3|^{-2} = |\theta_2|^{-2} \simeq 0.754877$. *The ordinary generating functions of the sequences* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *and* $\{R_n^2\}$, $\{R_{n+1}R_n\}$, $\{R_{n+2}R_n\}$ *are given as follows:*

(a)

$$
\begin{aligned}
\sum_{n=0}^{\infty} K_n^2 z^n &= \frac{z^4-z^3-z}{-z^6+z^5+z^4+z^3-z^2+z-1}, \\
\sum_{n=0}^{\infty} S_n^2 z^n &= \frac{4z^4+5z^3-9z^2+8z-9}{-z^6+z^5+z^4+z^3-z^2+z-1}, \\
\sum_{n=0}^{\infty} R_n^2 z^n &= \frac{z^5-2z^3-2z^2+z-1}{-z^6+z^5+z^4+z^3-z^2+z-1}.
\end{aligned}
$$

(b)

$$\begin{aligned}\sum_{n=0}^{\infty} K_{n+1}K_n z^n &= \frac{z}{-z^6+z^5+z^4+z^3-z^2+z-1},\\ \sum_{n=0}^{\infty} S_{n+1}S_n z^n &= \frac{6z^3-2z+3}{-z^6+z^5+z^4+z^3-z^2+z-1},\\ \sum_{n=0}^{\infty} R_{n+1}R_n z^n &= \frac{-z^5+2z^3+2z^2}{-z^6+z^5+z^4+z^3-z^2+z-1}.\end{aligned}$$

(c)

$$\begin{aligned}\sum_{n=0}^{\infty} K_{n+2}K_n z^n &= \frac{z^2-z}{-z^6+z^5+z^4+z^3-z^2+z-1},\\ \sum_{n=0}^{\infty} S_{n+2}S_n z^n &= \frac{6z^4-6z^3-2z^2+5z-3}{-z^6+z^5+z^4+z^3-z^2+z-1},\\ \sum_{n=0}^{\infty} R_{n+2}R_n z^n &= \frac{z^4-z^3-z^2-z+1}{-z^6+z^5+z^4+z^3-z^2+z-1}.\end{aligned}$$

From the last corollary, we obtain the following results for special cases of z.

Corollary 8.7. *Some infinite sums of* $\{K_n^2\}$, $\{K_{n+1}K_n\}$, $\{K_{n+2}K_n\}$ *and* $\{S_n^2\}$, $\{S_{n+1}S_n\}$, $\{S_{n+2}S_n\}$ *and* $\{R_n^2\}$, $\{R_{n+1}R_n\}$, $\{R_{n+2}R_n\}$ *are given as follows:*

(a) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{K_n^2}{2^n} &= \frac{36}{35},\\ \sum_{n=0}^{\infty} \frac{S_n^2}{2^n} &= \frac{408}{35},\\ \sum_{n=0}^{\infty} \frac{R_n^2}{2^n} &= \frac{78}{35}.\end{aligned}$$

(b) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{K_{n+1}K_n}{2^n} &= -\frac{32}{35},\\ \sum_{n=0}^{\infty} \frac{S_{n+1}S_n}{2^n} &= -\frac{176}{35},\\ \sum_{n=0}^{\infty} \frac{R_{n+1}R_n}{2^n} &= -\frac{46}{35}.\end{aligned}$$

(c) $z = \frac{1}{2}$.

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{K_{n+2}K_n}{2^n} &= \frac{16}{35},\\ \sum_{n=0}^{\infty}\frac{S_{n+2}S_n}{2^n} &= \frac{88}{35},\\ \sum_{n=0}^{\infty}\frac{R_{n+2}R_n}{2^n} &= -\frac{12}{35}.\end{aligned}$$

9. Generalized co-Padovan Numbers by Matrix Methods

In this section, we present matrix representations of the sequence Y_n, K_n and S_n. We also introduce Simson matrix and investigate its properties.

9.1. *Matrix Representations of the Sequences* $\boldsymbol{Y_n, K_n, S_n}$ *and* $\boldsymbol{R_n}$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Some properties of matrix A^n can be given as

$$\begin{aligned}A^n &= -A^{n-1} + tA^{n-3},\\ A^{n+m} &= A^nA^m = A^mA^n,\end{aligned}$$

for all integers m and n. Note that we have the following formulas:

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} Y_{n+1} \\ Y_n \\ Y_{n-1} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{n+2} \\ Y_{n+1} \\ Y_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n\begin{pmatrix} Y_2 \\ Y_1 \\ Y_0 \end{pmatrix},$$

and

$$\begin{pmatrix} K_{n+2} \\ K_{n+1} \\ K_n \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} K_{n+1} \\ K_n \\ K_{n-1} \end{pmatrix}.$$

We also define

$$B_n = \begin{pmatrix} K_{n+1} & K_{n-1} & K_n \\ K_n & K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-3} & K_{n-2} \end{pmatrix}$$

and

$$D_n = \begin{pmatrix} Y_{n+1} & Y_{n-1} & Y_n \\ Y_n & Y_{n-2} & Y_{n-1} \\ Y_{n-1} & Y_{n-3} & Y_{n-2} \end{pmatrix}.$$

Theorem 9.1. *For all integers m, n, we have the following properties:*

(a) $B_n = A^n$, *i.e.,*

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} K_{n+1} & K_{n-1} & K_n \\ K_n & K_{n-2} & K_{n-1} \\ K_{n-1} & K_{n-3} & K_{n-2} \end{pmatrix}.$$

(b) $D_1 A^n = A^n D_1$.
(c) $D_{n+m} = D_n B_m = B_m D_n$, *i.e.,*

$$\begin{aligned} &\begin{pmatrix} Y_{n+m+1} & Y_{n+m-1} & Y_{n+m} \\ Y_{n+m} & Y_{n+m-2} & Y_{n+m-1} \\ Y_{n+m-1} & Y_{n+m-3} & Y_{n+m-2} \end{pmatrix} \\ = &\begin{pmatrix} Y_{n+1} & Y_{n-1} & Y_n \\ Y_n & Y_{n-2} & Y_{n-1} \\ Y_{n-1} & Y_{n-3} & Y_{n-2} \end{pmatrix} \begin{pmatrix} K_{m+1} & K_{m-1} & K_m \\ K_m & K_{m-2} & K_{m-1} \\ K_{m-1} & K_{m-3} & K_{m-2} \end{pmatrix} \\ = &\begin{pmatrix} K_{m+1} & K_{m-1} & K_m \\ K_m & K_{m-2} & K_{m-1} \\ K_{m-1} & K_{m-3} & K_{m-2} \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_{n-1} & Y_n \\ Y_n & Y_{n-2} & Y_{n-1} \\ Y_{n-1} & Y_{n-3} & Y_{n-2} \end{pmatrix}. \end{aligned}$$

(d)

$$A^n = K_{n-1}A^2 + K_{n-3}A + K_{n-2}I,$$

i.e.,
$A^n = (K_{n+2} + K_{n+1})A^2 + (K_{n+2} + K_{n+1} + K_n)A + (K_{n+1} + K_n)I$
that is,
$A^n = K_{n+2}(A^2 + A) + K_{n+1}(A^2 + A + I) + K_n(A + I)$
where

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Proof. Set $W_n = Y_n$, $G_n = K_n$ and $r = -1,\ s = 0,\ t = 1$ in [4, Theorem 51.]. $\square$

Next, we present matrix formulas for the generalized co-Padovan,co-Perrin (co-Padovan-Lucas) and co-Padovan numbers.

Corollary 9.2. *For all integers n, we have the following formulas for generalized co-Padovan,co-Perrin (co-Padovan-Lucas) and co-Padovan numbers.*

(a) *Generalized co-Padovan numbers.*

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{\Lambda_Y(0)} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+3} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n+2} + (Y_1^2 - Y_0Y_2)Y_{n+1} \\
a_{21} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+2} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n \\
a_{31} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+1} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1} \\
a_{12} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+1} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1} \\
a_{22} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_n + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2} \\
a_{32} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n-1} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n-2} + (Y_1^2 - Y_0Y_2)Y_{n-3} \\
a_{13} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+2} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n+1} + (Y_1^2 - Y_0Y_2)Y_n \\
a_{23} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_{n+1} + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_n + (Y_1^2 - Y_0Y_2)Y_{n-1} \\
a_{33} &= (-Y_1^2 + Y_0^2 - Y_1Y_2)Y_n + (Y_2^2 + Y_1Y_2 - Y_0Y_1)Y_{n-1} + (Y_1^2 - Y_0Y_2)Y_{n-2}
\end{aligned}$$

and

$$\Lambda_Y(0) = Y_2^3 + Y_1^3 + Y_0^3 + 2Y_1Y_2^2 + Y_2Y_1^2 - Y_0Y_1^2 - Y_0^2Y_2 - 3Y_2Y_1Y_0.$$

(b) *co-Perrin (co-Padovan-Lucas) numbers.*

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \frac{1}{23} \begin{pmatrix} 9S_{n+3} + 3S_{n+2} - 2S_{n+1} & 9S_{n+1} + 3S_n - 2S_{n-1} & 9S_{n+2} + 3S_{n+1} - 2S_n \\ 9S_{n+2} + 3S_{n+1} - 2S_n & 9S_n + 3S_{n-1} - 2S_{n-2} & 9S_{n+1} + 3S_n - 2S_{n-1} \\ 9S_{n+1} + 3S_n - 2S_{n-1} & 9S_{n-1} + 3S_{n-2} - 2S_{n-3} & 9S_n + 3S_{n-1} - 2S_{n-2} \end{pmatrix}$$

(c) *co-Padovan numbers.*

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n = \begin{pmatrix} R_{n+3} + R_{n+2} + R_{n+1} & R_{n+1} + R_n + R_{n-1} & R_{n+2} + R_{n+1} + R_n \\ R_{n+2} + R_{n+1} + R_n & R_n + R_{n-1} + R_{n-2} & R_{n+1} + R_{n-1} + R_n \\ R_{n+1} + R_n + R_{n-1} & R_{n-1} + R_{n-2} + R_{n-3} & R_n + R_{n-1} + R_{n-2} \end{pmatrix}.$$

Proof. Set $W_n = Y_n, r = -1,\ s = 0,\ t = 1$ and then $Y_n = K_n$ and $Y_n = R_n$, respectively, in [4, Corollary 52.]. □

Now, we present an identity for Y_{n+m}.

Theorem 9.3. *(Honsberger's Identity) For all integers m and n, we have*

$$\begin{aligned} Y_{n+m} &= Y_nK_{m+1} + Y_{n-1}K_{m-1} + Y_{n-2}K_m \\ &= Y_nK_{m+1} + Y_{n-2}K_m + Y_{n-1}K_{m-1}. \end{aligned}$$

Proof. Set $W_n = Y_n$, $r = -1,\ s = 0,\ t = 1$ and then $G_n = K_n$ in [4, Theorem 53.].
□

As special cases of the last Theorem, we have the following corollary.

Corollary 9.4. *For all integers m, n, we have the following properties:*

$$\begin{array}{rcl} K_{n+m} & = & K_n K_{m+1} + K_{n-1} K_{m-1} + K_{n-2} K_m, \\ S_{n+m} & = & S_n K_{m+1} + S_{n-1} K_{m-1} + S_{n-2} K_m, \\ Y_{n+m} & = & Y_n K_{m+1} + Y_{n-1} K_{m-1} + Y_{n-2} K_m \end{array}$$

Next, we present identies for Y_{mn+j} and its special cases.

Corollary 9.5. *For all integers m, n, j, we have the following properties:*

$$\begin{array}{rcl} Y_{mn+j} & = & K_{mn-1} Y_{j+2} + K_{mn-3} Y_{j+1} + K_{mn-2} Y_j, \\ K_{mn+j} & = & K_{mn-1} K_{j+2} + K_{mn-3} K_{j+1} + K_{mn-2} K_j, \\ S_{mn+j} & = & K_{mn-1} S_{j+2} + K_{mn-3} S_{j+1} + K_{mn-2} S_j, \\ R_{mn+j} & = & K_{mn-1} R_{j+2} + K_{mn-3} R_{j+1} + K_{mn-2} R_j \end{array}$$

Proof. Set $r = -1$, $s = 0$, $t = 1$ and $W_n = Y_n$, then take $Y_n = K_n$, $Y_n = S_n$ and $Y_n = R_n$, respectively,in [4, Corollary 55.]. □

9.2. *Simson Matrix and its Properties*

For $n \in \mathbb{Z}$, we define

$$f_Y(n) = \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix}.$$

We call this matrix as Simson matrix of the sequence Y_n. Similarly, as special cases of Y_n, Simson matrices of the sequences K_n and S_n are

$$f_K(n) = \begin{pmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{pmatrix} \text{ and } f_S(n) = \begin{pmatrix} S_{n+2} & S_{n+1} & S_n \\ S_{n+1} & S_n & S_{n-1} \\ S_n & S_{n-1} & S_{n-2} \end{pmatrix}$$

,

respectively.

Lemma 9.5.1. *For all integers n, m and j, the followings hold.*

(a) $f_Y(n) = r f_Y(n-1) + s f_Y(n-2) + t f_Y(n-3)$.
(b) $f_Y(n) = A f_Y(n-1)$ *and* $f_Y(n) = A^n f_Y(0)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \\ Y_{n-1} & Y_{n-2} & Y_{n-3} \end{pmatrix}$$

and

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{pmatrix}.$$

(c) $f_Y(n+m) = A^n f_Y(m)$ *and* $f_Y(n+m) = A^m f_Y(n)$ *i.e.,*

$$\begin{pmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{pmatrix},$$

and

$$\begin{pmatrix} Y_{m+n+2} & Y_{m+n+1} & Y_{m+n} \\ Y_{m+n+1} & Y_{m+n} & Y_{m+n-1} \\ Y_{m+n} & Y_{m+n-1} & Y_{m+n-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix},$$

and $f_Y(n) = A^m f_Y(n-m)$, *i.e.,*

$$\begin{pmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^m \begin{pmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{pmatrix}.$$

Proof. Set $W_n = Y_n$ and $r = -1,\ s = 0,\ t = 1$ in [4 , Lemma 56.]. □

Taking the determinant of both sides of the identities given in the last Lemma, we obtain the following Theorem.

Theorem 9.6. *For all integers* n *and* m, *the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_Y(n+m)) = \det(f_Y(m)) \text{ and } \det(f_Y(n)) = \det(f_Y(n-m)),$$

i.e.,

$$\begin{vmatrix} Y_{n+m+2} & Y_{n+m+1} & Y_{n+m} \\ Y_{n+m+1} & Y_{n+m} & Y_{n+m-1} \\ Y_{n+m} & Y_{n+m-1} & Y_{n+m-2} \end{vmatrix} = \begin{vmatrix} Y_{m+2} & Y_{m+1} & Y_m \\ Y_{m+1} & Y_m & Y_{m-1} \\ Y_m & Y_{m-1} & Y_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_{n-m+2} & Y_{n-m+1} & Y_{n-m} \\ Y_{n-m+1} & Y_{n-m} & Y_{n-m-1} \\ Y_{n-m} & Y_{n-m-1} & Y_{n-m-2} \end{vmatrix}.$$

(b) *(see Theorem 5.1) Simson's (or Cassini's) Identity:*

$$\det(f_Y(n)) = \det(f_Y(0)),$$

i.e.,

$$\begin{vmatrix} Y_{n+2} & Y_{n+1} & Y_n \\ Y_{n+1} & Y_n & Y_{n-1} \\ Y_n & Y_{n-1} & Y_{n-2} \end{vmatrix} = \begin{vmatrix} Y_2 & Y_1 & Y_0 \\ Y_1 & Y_0 & Y_{-1} \\ Y_0 & Y_{-1} & Y_{-2} \end{vmatrix}.$$

Proof. Set $W_n = Y_n$, and $r = -1,\ s = 0,\ t = 1$ in [4, Theorem 57.]. □

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-adjusted Padovan numbers (take $Y_n = K_n$ with $K_0 = 0, K_1 = 1, K_2 = -1$).

Corollary 9.7. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_K(n+m)) = \det(f_K(m)) \quad \text{and} \quad \det(f_K(n)) = \det(f_K(n-m)),$$

i.e.,

$$\begin{vmatrix} K_{n+m+2} & K_{n+m+1} & K_{n+m} \\ K_{n+m+1} & K_{n+m} & K_{n+m-1} \\ K_{n+m} & K_{n+m-1} & K_{n+m-2} \end{vmatrix} = \begin{vmatrix} K_{m+2} & K_{m+1} & K_m \\ K_{m+1} & K_m & K_{m-1} \\ K_m & K_{m-1} & K_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = \begin{vmatrix} K_{n-m+2} & K_{n-m+1} & K_{n-m} \\ K_{n-m+1} & K_{n-m} & K_{n-m-1} \\ K_{n-m} & K_{n-m-1} & K_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_K(n)) = \det(f_K(0)),$$

i.e.,

$$\begin{vmatrix} K_{n+2} & K_{n+1} & K_n \\ K_{n+1} & K_n & K_{n-1} \\ K_n & K_{n-1} & K_{n-2} \end{vmatrix} = -1.$$

Taking $Y_n = S_n$ with $S_0 = 3, S_1 = -1, S_2 =$ in the last Theorem, we have the following Corollary which gives determinantal formulas of co-Perrin (co-Padovan-Lucas) numbers.

Corollary 9.8. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_S(n+m)) = t^n \det(f_S(m)) \quad \text{and} \quad \det(f_S(n)) = t^m \det(f_S(n-m))$$

i.e.,

$$\begin{vmatrix} S_{n+m+2} & S_{n+m+1} & S_{n+m} \\ S_{n+m+1} & S_{n+m} & S_{n+m-1} \\ S_{n+m} & S_{n+m-1} & S_{n+m-2} \end{vmatrix} = \begin{vmatrix} S_{m+2} & S_{m+1} & S_{m} \\ S_{m+1} & S_{m} & S_{m-1} \\ S_{m} & S_{m-1} & S_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = \begin{vmatrix} S_{n-m+2} & S_{n-m+1} & S_{n-m} \\ S_{n-m+1} & S_{n-m} & S_{n-m-1} \\ S_{n-m} & S_{n-m-1} & S_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_S(n)) = \det(f_S(0)),$$

i.e.,

$$\begin{vmatrix} S_{n+2} & S_{n+1} & S_{n} \\ S_{n+1} & S_{n} & S_{n-1} \\ S_{n} & S_{n-1} & S_{n-2} \end{vmatrix} = -23.$$

From the last Theorem, we have the following Corollary which gives determinantal formulas of co-Padovan numbers (take $Y_n = R_n$ with $R_0 = 1, R_1 = 0, R_2 = -1$).

Corollary 9.9. *For all integers n and m, the following identities hold.*

(a) *Catalan's Identity:*

$$\det(f_R(n+m)) = \det(f_R(m)) \quad \textit{and} \quad \det(f_R(n)) = \det(f_R(n-m)),$$

i.e.,

$$\begin{vmatrix} R_{n+m+2} & R_{n+m+1} & R_{n+m} \\ R_{n+m+1} & R_{n+m} & R_{n+m-1} \\ R_{n+m} & R_{n+m-1} & R_{n+m-2} \end{vmatrix} = \begin{vmatrix} R_{m+2} & R_{m+1} & R_{m} \\ R_{m+1} & R_{m} & R_{m-1} \\ R_{m} & R_{m-1} & R_{m-2} \end{vmatrix},$$

and

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_{n} \\ R_{n+1} & R_{n} & R_{n-1} \\ R_{n} & R_{n-1} & R_{n-2} \end{vmatrix} = \begin{vmatrix} R_{n-m+2} & R_{n-m+1} & R_{n-m} \\ R_{n-m+1} & R_{n-m} & R_{n-m-1} \\ R_{n-m} & R_{n-m-1} & R_{n-m-2} \end{vmatrix}.$$

(b) *Simson's (or Cassini's) Identity:*

$$\det(f_R(n)) = \det(f_R(0)),$$

i.e.,

$$\begin{vmatrix} R_{n+2} & R_{n+1} & R_{n} \\ R_{n+1} & R_{n} & R_{n-1} \\ R_{n} & R_{n-1} & R_{n-2} \end{vmatrix} = -1$$

References

[1] A. G. Shannon, A.G., Anderson, P.G., Horadam A.F., Properties of Cordonnier, Perrin and Van der Laan numbers, International Journal of Mathematical Education in Science and Technology, 37(7), 825–831, 2006.

[2] Sloane, N.J.A., The on-line encyclopedia of integer sequences, http://oeis.org/

[3] Soykan, Y., On Generalized Padovan Numbers, International Journal of Advances in Applied Mathematics and Mechanics, 10(4), 72-90, 2023.

[4] Soykan, Y., Generalized Tribonacci Polynomials, Earthline Journal of Mathematical Sciences, 13(1), 1-120, 2023. https://doi.org/10.34198/ejms.13123.1120

[5] Soykan, Y., Sums and Generating Functions of Squares of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$, International Journal of Mathematics, Statistics and Operations Research, 3(2), 281-300, 2023. https://doi.org/10.47509/IJMSOR.2023.v03i02.06

[6] Soykan, Y., Sums and Generating Functions of Squares of Special Cases of Generalized Tribonacci Polynomials: Closed Formulas of $\sum_{k=0}^{n} z^k W_k^2$ and $\sum_{n=0}^{\infty} W_n^2 z^n$, International Journal of Advances in Applied Mathematics and Mechanics, 12(3), 1-72, 2025.